

Surface Holonomy

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1 Introduction

From the theory of hermitian line bundles with connection we single out one interesting aspect: such a line bundle $L \rightarrow M$ assigns to each loop $\gamma : S^1 \rightarrow M$ a certain complex number $\text{hol}_L(\gamma)$ in $U(1)$ which is called the holonomy of L around γ . This assignment has many properties, and we will gradually encounter some of them during this article. It is worthwhile to specify three of them here, which will turn out to be analogous for surface holonomy. To start with, the holonomy of L measures the curvature of L in the following way: for an embedded two-dimensional submanifold D of M with parameterized boundary ∂D one finds

$$\exp\left(\int_D \text{curv}(L)\right) = \text{hol}_L(\partial D).$$

This equality can be seen as an improvement of Stokes' Theorem in two dimensions,

$$\int_D F = \int_{\partial D} \rho$$

which expresses the integral of the closed 2-form F by something on the boundary, just like the above formula for the integral of the curvature of the line bundle L . However, Stokes' Theorem is restricted to exact 2-forms $F = d\rho$ whose cohomology class $[F] \in H^2(M, \mathbb{R})$ vanishes. Our improvement allows us at the price of exponentiation not only to take 2-forms with trivial cohomology class, but also – and more general – 2-forms whose class lies in the image of the homomorphism $H^k(M, \mathbb{Z}) \rightarrow H^k(M, \mathbb{R})$. Those 2-forms F – which we shall call forms with integral class – arise as the curvature of a hermitian line bundle L with connection. Then,

$$\int_D F = \ln(\text{hol}_L(\partial D)) \pmod{\mathbb{Z}}$$

generalizes Stokes' Theorem.

A second reason why holonomy is interesting lies in understanding the assignment $\gamma \mapsto \text{hol}_L(\gamma)$ as a $U(1)$ -valued function $\text{hol}_L : LM \rightarrow U(1)$ on the loop space LM that consists of all smooth loops $S^1 \rightarrow M$. The loop space can canonically be endowed with the structure of an infinite dimensional manifold, so that the map hol_L becomes smooth [Bry93]. One can consider going in the other direction, when one is concerned with a smooth $U(1)$ -valued function on the loop space and is able to express it as the holonomy of a hermitian line bundle with connection over the – finite dimensional – manifold M .

Also of importance is the application of line bundles in physics: a hermitian line bundle $L \rightarrow M$ with connection offers a natural description of a $U(1)$ -gauge theory. On the quantum level, such a theory can be defined by assigning a complex number to each particle moving (on a closed line) through the target space M ; this number acts as an amplitude in some path integral. If a particle moves on a circle $\gamma : S^1 \rightarrow M$, its amplitude is given by

$$\mathcal{A}(\gamma) = e^{S_{\text{kin}}(\gamma)} \cdot \text{hol}_L(\gamma),$$

where $S_{\text{kin}}(\gamma)$ is a kinetic term, and the holonomy expresses the coupling to the gauge field. The line bundle L comes up with all features you would expect from a gauge field: its curvature is a 2-form F and may be called the field strength of the gauge field. The second Bianchi identity $dF = 0$ coincides with one of Maxwell's equations, and – last but not least – the fact that F has an integral class is nothing but Dirac's quantization condition for the electric charge.

The question for an appropriate concept of surface holonomy has similar origins:

- it could provide a generalization Stokes' Theorem in three dimensions similar to that in two dimensions indicated above.
- it could be used in string theory to couple strings to non-trivial background fluxes, analogous to the coupling of a point particle to a gauge field.
- it could also provide actions for 3-form gauge fields in certain (classical) gauge theories.
- it could provide a way to describe structure on the loop space by structure on a finite-dimensional space, very much in the same way like smooth $U(1)$ -valued functions may be interpreted as the holonomy of a hermitian line bundle with connection over M .

Let us pick out the first two points and describe how they are related to surface holonomy. The generalization of Stokes' Theorem is obvious: if H is a 3-form on a three-dimensional manifold B , which may be not exact but with integral class, its integral over B could be expressed as (the logarithm of) the holonomy around the surface $\Sigma = \partial B$.

In fact, this is exactly a question which arises in two-dimensional conformal field theory, when studying non-linear sigma models on a Lie group G . Such a model can be defined by amplitudes $\mathcal{A}(\phi)$ for some path integral, where ϕ is a map from a closed complex surface Σ – the worldsheet – into the target space G of the model. In [Wit84], Witten gives the following definition for $G = SU(2)$. Σ is the boundary of a three dimensional manifold B , and because the homotopy groups $\pi_i(SU(2))$ vanish for $i = 1, 2$, every map $\phi : \Sigma \rightarrow M$ can be extended into the interior B to a map $\Phi : B \rightarrow G$. From the theory of compact, simple, connected and simply connected Lie groups it is known that the Ad-invariant trilinear form $k\langle -, [-, -] \rangle$ on the Lie algebra $\mathfrak{su}(2)$ induces a closed, bi-invariant 3-form H on $SU(2)$ with integral class, provided k is an integer. Witten showed that – due to the integrality of H –

$$\mathcal{A}(\phi) := \exp \left(S_{\text{kin}}(\phi) + \int_B \Phi^* H \right)$$

neither depends on the choice of B nor on the choice of the extension Φ , so that he obtained a well-defined amplitude. Here $S_{\text{kin}}(\phi)$ is a kinetic term, and with a certain relative normalization this model is called the Wess-Zumino-Witten model on G at level k .

Now, if we could express the integral of $\Phi^* H$ over B by something on the boundary Σ – for instance via a generalization of Stokes' Theorem – the definition of the amplitude $\mathcal{A}(\phi)$ would be independent of conditions on the existence of the extension Φ , in particular of the simply-connectedness of G . It would hence provide a proper definition of Wess-Zumino-Witten models on arbitrary simple and compact Lie groups.

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2 Gerbes

To define surface holonomy, we first need a mathematical object which plays the role of the hermitian line bundle with connection. Such an object is collectively called gerbe with connective structure.

What makes gerbes a bit mysterious is that there are numerous definitions which look outmost different. To give an impression we show some examples of the different manifestations of gerbes and their connective structures.

Gerbes as stacks. This is the original definition given in 1971 by J. Giraud with a view to non-abelian cohomology [Gir71]. A stack is a fibred category satisfying a gluing (or descent) axiom. According to Giraud, a gerbe is a stack, whose fibres are groupoids and satisfy a certain transitivity and non-emptiness condition, also see [Moe02]. As an important characterization of a gerbe, Giraud defines the band of a gerbe, a certain sheaf of groups.

The definition of a connective structure on a gerbe in this sense was given thirty years later by L. Breen and W. Messing [BM05], who considered gerbes with arbitrary bands living over a scheme.

Gerbes as Cohomology Classes. Around 1972, P. Deligne invented a cohomology theory which is now called Deligne cohomology [Del91, Bry93]. It is build up on cochain complexes

$$0 \longrightarrow \mathcal{D}^0(n) \xrightarrow{\mathcal{D}} \mathcal{D}^1(n) \xrightarrow{\mathcal{D}} \dots \xrightarrow{\mathcal{D}} \mathcal{D}^k(n) ,$$

one for each natural number n . Deligne was originally interested in algebraic geometry, and realized that the cohomology group $H^1(M, \mathcal{D}(1))$ classifies hermitian line bundles with connection [Del91].

A class in the cohomology group $H^2(M, \mathcal{D}(2))$ can be seen as a $U(1)$ -banded gerbe with connective structure in the sense that it provides a definition of holonomy around surfaces. This was shown by K. Gawędzki and applied to topological field theory [Gaw88]. Gawędzki also showed that gerbes are related to structure on the loop space, namely to a line bundle, which is obtained by a so-called transgression procedure.

Deligne cohomology provides a natural way to see gerbes with connective structure in a hierarchy of objects, starting with $U(1)$ -valued functions, hermitian line bundles with connection and gerbes with connective structure, and which is continued by n -gerbes which are classified by $H^{n+1}(M, \mathcal{D}(n+1))$.

Gerbes as Sheaves of Groupoids. This concept is strongly related to the original one of a certain kind of stack, and defines a gerbe over a manifold M by an assignment

$$U \mapsto \mathcal{G}(U)$$

of a groupoid $\mathcal{G}(U)$ to any open subset U of M . It is developed in great detail in J.-L. Brylinski's book [Bry93]. Analogous to the gluing axiom of a sheaf, gluing axioms for such a gerbe are given. For \mathbb{C}^\times -banded sheaves of groupoids, Brylinski develops the definition and properties of a connective structure. He gives precise relations and classification result between sheaves of groupoids and Deligne cohomology. Furthermore he constructs a line bundle over the loop space and shows that it coincides with the one Gawędzki constructed from Deligne cohomology.

Gerbes as Bundle Gerbes. The concept of a sheaf of groupoids and even more the one of a connective structure on it is quite general but also quite complex. For some purposes, e.g. for the definition of holonomy and for applications in conformal field theory, it is sufficient to use a simplified version – simplified in the sense that it just uses well-known geometric structure like line bundles and differential forms. For $U(1)$ - or \mathbb{C}^\times -banded gerbes, M. K. Murray invented bundle gerbes [Mur96], which we will use in this article to develop surface holonomy. As we will learn, bundle gerbes admit a very simple and natural definition of a connective structure. To make contact to other concepts of gerbe, we will see, that every bundle gerbe induces a sheaf of groupoids, and that their isomorphism classes are in bijection to the Deligne cohomology group $H^2(M, \mathcal{D}(2))$. Bundle gerbes have also been used in two-dimensional conformal field theory [GR02, Gaw05, SSW05].

Gerbes defined on open Covers. Closely related to bundle gerbes are gerbes defined on open covers [Hit01], although bundle gerbes are a bit more general.

Gerbes as 2-Bundles. A somewhat different approach to surface holonomy is by categorification of a vector bundle with connection. This leads to the concept of a 2-bundle with certain additional structure [BS04]. This approach also covers gerbes with non-abelian bands. Furthermore it realizes consequently the 2-categorical nature of gerbes, which we will also discover during this article.

For the purposes of this article we consider gerbes with band $U(1)$, also called abelian (hermitian) gerbes. Accordingly we drop the qualifier hermitian for gerbes and for line bundles to improve the readability.

3 From Line Bundles to Bundle Gerbes

As indicated before, bundle gerbes are built up of line bundles and differential forms. One of basic features of a line bundle $L \rightarrow M$ is that it is locally trivializable. This is usually stated with respect to an open cover, but here we state it with respect to a covering $\pi : Y \rightarrow M$. From any open cover $\{V_i\}_{i \in I}$ of M one can produce such a covering by defining Y as the disjoint union of the sets,

$$Y = \bigsqcup_{i \in I} V_i,$$

and π as patched together from the inclusions $V_i \hookrightarrow M$. Locally trivializable means that there is a covering $\pi : Y \rightarrow M$ and a commutative diagram

$$\begin{array}{ccc} Y \times \mathbb{C} & \longrightarrow & L \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\pi} & M. \end{array}$$

A local trivialization defines a transition function $g : Y^{[2]} \rightarrow U(1)$, where $Y^{[k]}$ denotes the k -fold fibre product of Y with itself. The transition function satisfies the cocycle condition

$$\pi_{12}^* g \cdot \pi_{23}^* g = \pi_{13}^* g \tag{1}$$

over $Y^{[3]}$, where $\pi_{12} : Y^{[3]} \rightarrow Y^{[2]}$ are the projections on the respective components. If Y comes from an open cover, $Y^{[k]}$ is the disjoint union of all k -fold intersections of the open sets V_i . Accordingly, the transition function g decomposes in functions $g_{ij} : V_i \cap V_j \rightarrow U(1)$, and the cocycle condition becomes $g_{ij} \cdot g_{jk} = g_{ik}$ as functions on $V_i \cap V_j \cap V_k$.

Bundle gerbes don't have a total space as line bundles do. For the definition of a bundle gerbe we step in after having locally trivialized, i.e. after having chosen a covering $\pi : Y \rightarrow M$,

$$\begin{array}{ccc} & ? & \\ & \downarrow & \\ Y & \xrightarrow{\pi} & M. \end{array}$$

Now we define the bundle gerbe analogous to what remains of the locally trivialized line bundle. According to our remarks about Deligne cohomology, $U(1)$ -valued functions, hermitian line bundles with connection and gerbes with connective structure form a certain series of objects. Now we move up one step: instead of a transition function on $Y^{[2]}$, we take a line bundle

$L \rightarrow Y^{[2]}$. The next steps are predicted: because we can't multiply line bundles like the pullbacks of the transition function g in (1), the cocycle condition has to be relaxed to an isomorphism

$$\mu : \pi_{12}^* L \otimes \pi_{23}^* L \rightarrow \pi_{13}^* L$$

of line bundles over $Y^{[3]}$ – called the groupoid multiplication. To capture an essential aspect of the multiplication of functions, we demand that this isomorphism is associative.

It is straightforward to define a connective structure on a bundle gerbe. Consider first a connection on a line bundle $L \rightarrow M$. In a local trivialization $\pi : Y \rightarrow M$, this connection defines a 1-form $A \in \Omega^1(Y)$, which is related to the transition function g by

$$g^{-1} dg = \pi_2^* A - \pi_1^* A.$$

For the bundle gerbe, we take a connection on the line bundle $L \rightarrow Y^{[2]}$ and impose that the isomorphism μ respects connections. Additionally, we take a 2-form $C \in \Omega^2(Y)$ – called the curving – which has to be related to the connection on L by

$$\text{curv}(L) = \pi_2^* C - \pi_1^* C.$$

The connection on L together with the curving C form the connective structure. It is shown in [Mur96] that every bundle gerbe admits a connective structure.

In the rest of this article, we will only be concerned with bundle gerbes with connective structure, and hence drop the last suffix. We will also understand a line bundle as a (hermitian) line bundle with connection. Accordingly, all isomorphisms of line bundles will be isomorphisms of line bundles which preserve the connections. With these conventions, we are arrived at the following definition:

Definition 1. *A bundle gerbe \mathcal{G} over M consists of a covering $\pi : Y \rightarrow M$, a 2-form $C \in \Omega^2(Y)$, a line bundle $L \rightarrow Y^{[2]}$ and an isomorphism*

$$\mu : \pi_{12}^* L \otimes \pi_{23}^* L \rightarrow \pi_{13}^* L$$

of line bundles over $Y^{[3]}$. Two axioms have to be satisfied:

(G1) *the curvature of L is related to the curving by*

$$\text{curv}(L) = \pi_2^* C - \pi_1^* C.$$

(G2) the groupoid multiplication is associative in the sense that the diagram

$$\begin{array}{ccc}
\pi_{12}^* L \otimes \pi_{23}^* L \otimes \pi_{34}^* L & \xrightarrow{\pi_{123}^* \mu \otimes 1} & \pi_{13}^* L \otimes \pi_{34}^* L \\
\downarrow 1 \otimes \pi_{234}^* \mu & & \downarrow \pi_{134}^* \mu \\
\pi_{12}^* L \otimes \pi_{24}^* L & \xrightarrow{\pi_{124}^* \mu} & \pi_{14}^* L
\end{array}$$

of isomorphisms of line bundles over $Y^{[4]}$ is commutative.

In the following we will specify some properties of this definition and show several constructions what one can do with it. Similar to line bundles (with connection), each bundle gerbe \mathcal{G} determines a closed 3-form $\text{curv}(\mathcal{G})$ on M , called the curvature of \mathcal{G} : the derivative of axiom (G1) gives $\pi_1^* dC = \pi_2^* dC$, since the curvature of the line bundle L is a closed form. This means that dC descends along $\pi : Y \rightarrow M$ to a 3-form on M – the curvature of the bundle gerbe \mathcal{G} . It is obviously a closed form, and it will turn out later that it has an integral class.

To give an example of a bundle gerbe, we introduce trivial bundle gerbes. Just as for every 1-form $A \in \Omega^1(M)$ there is a trivial line bundle over M having this 1-form as its connection, we find a trivial bundle gerbe for every 2-form $\rho \in \Omega^2(M)$. The construction of this bundle gerbe is quite easy: for the covering we take the identity $\text{id} : M \rightarrow M$, and the curving is the given 2-form ρ . The line bundle L is the trivial line bundle with the trivial connection, and the groupoid multiplication is the identity isomorphism between trivial line bundles. Now, axiom (G1) is satisfied since $\text{curv}(L) = 0$ and $\pi_1 = \pi_2 = \text{id}_M$. The associativity axiom (G2) is surely satisfied by the identity isomorphism. Thus we have defined a bundle gerbe, which we denote by \mathcal{I}_ρ . The curvature of a trivial gerbe is $\text{curv}(\mathcal{I}_\rho) = d\rho$.

Less elementary examples of bundle gerbes have been constructed in [GR02, Mei02, GR03], namely all (bi-invariant) bundle gerbes over all simple compact Lie groups. The availability of concrete examples in such non-trivial cases is an important advantage of bundle gerbes. Their constructions were possible by an explicit use of the geometric nature of bundle gerbes. For example, to construct a bundle gerbe over $SU(N)$, the line bundle L occurs as the canonical line bundle on the coadjoint orbit through a simple weight of some representation of $SU(N)$. Another remarkable aspect is that the Lie-theoretic construction of bundle gerbes over Lie groups apart from $SU(N)$ or $Sp(4n)$ makes use of the fact that a covering $\pi : Y \rightarrow M$ is more general than having an open cover of M : the dimension of Y may be greater than the dimension of M .

Let us again assume that the covering $\pi : Y \rightarrow M$ of a bundle gerbe \mathcal{G} comes from an open cover $\{V_i\}_{i \in I}$ of M (in fact this is the kind of gerbe which was called before a gerbe defined on open covers [Hit01]). Remember that we introduced the gerbe data – namely the line bundle L , the groupoid multiplication μ and the curving C – as analogues of the data of a trivialized line bundle. Surely the curving restricts to a 2-form B_i on each open set V_i . To get similar local expressions for the line bundle and the groupoid multiplication, we can trivialize once more: if the open sets V_i are chosen such that every double intersection $V_i \cap V_j$ is contractible, we are able to choose sections

$$\sigma_{ij} : V_i \cap V_j \rightarrow L$$

of unit length. Then, the connection on L pulls back to 1-forms A_{ij} on each double intersection $V_i \cap V_j$. Furthermore, over a three-fold intersection $V_i \cap V_j \cap V_k$, we can multiply two sections using the groupoid multiplication, and compare the result to a third section,

$$\mu(\sigma_{ij} \otimes \sigma_{jk}) = g_{ijk} \cdot \sigma_{ik},$$

via a function $g_{ijk} : V_i \cap V_j \cap V_k \rightarrow U(1)$. Summarizing, we have extracted $U(1)$ -valued functions g_{ijk} on three-fold intersections, 1-forms A_{ij} on two-fold intersections, and 2-forms B_i on each open set. One can deduce the following relations among this local data:

$$\begin{aligned} g_{ijk} \cdot g_{ikl} &= g_{jkl} \cdot g_{ijl} \\ g_{ijk}^{-1} dg_{ijk} &= A_{jk} - A_{ik} + A_{ij} \\ dA_{ij} &= B_j - B_i. \end{aligned}$$

The first one is a consequence of the associativity of μ from axiom (G2), the second is the fact that μ preserves connections, and the third is the curvature condition (G1). These equations look like analogues of the two conditions for local data of a line bundle – the transition functions $g_{ij} : V_i \cap V_j \rightarrow U(1)$ and the connection 1-forms A_i on V_i – namely

$$\begin{aligned} g_{ij} \cdot g_{jk} &= g_{ik} \\ g_{ij}^{-1} dg_{ij} &= A_j - A_i. \end{aligned}$$

A precise meaning of this analogy is given by Deligne cohomology. We already have indicated that Deligne cohomology comes from a cochain complex with cochain groups $\mathcal{D}^k(n)$ and a coboundary operator D . Their definition is such that the collection of local data (g, A, B) of the bundle gerbe \mathcal{G} defines an element in $\mathcal{D}^2(2)$. The three relations among this local data give the cocycle

condition $D(g, A, B) = 0$, so that we see, that the bundle gerbe \mathcal{G} defines a class in the cohomology group $H^2(M, \mathcal{D}(2))$. Similarly, the collection of local data (g, A) of a line bundle forms an element in $\mathcal{D}^1(1)$, and the two relations above give the cocycle condition $D(g, A) = 0$. So, a line bundle defines a class in $H^1(M, \mathcal{D}(1))$. In the next section we will see, that with the correct definition of isomorphisms between bundle gerbes, the correspondence of isomorphism classes of bundle gerbes with classes in Deligne cohomology is one-to-one.

To continue with specifying properties of a bundle gerbe, we come to the definition of characteristic classes. The transition function $g_{ij} : V_i \cap V_j \rightarrow U(1)$ of a line bundle $L \rightarrow M$ with respect to an open cover $\{V_i\}_{i \in I}$ defines a class $[g]$ in the Čech cohomology of the sheaf $\underline{U(1)}$ with respect to the chosen open cover. Via the exponential sequence, it hence gives rise to a class

$$[g] \in H^2(M, \mathbb{Z}),$$

which is independent of the cover and of the sections chosen to get the transition function. It is thus an intrinsic quantity of the line bundle L , called the (first) Chern class and denoted by $c_1(L)$. The image of the Chern class in de Rham cohomology is equal to the class of the curvature of L ,

$$c_1(L) = [\text{curv}(L)],$$

which proves that the curvature of a line bundle with connection is a 2-form with integral class.

In the same way, the function $g_{ijk} : V_i \cap V_j \cap V_k \rightarrow U(1)$ defined by the groupoid multiplication of a bundle gerbe \mathcal{G} trivialized on an open cover, defines a class

$$[g] \in H^3(M, \mathbb{Z})$$

which is also independent of the open cover and of the sections σ_{ij} which were chosen to extract the local data. This class is called the Dixmier-Douady class of the bundle gerbe \mathcal{G} and denoted by $dd(\mathcal{G})$. The Dixmier-Douady class and the curvature of \mathcal{G} obey the same relation as the Chern class and the curvature of a line bundle [Mur96]:

$$dd(\mathcal{G}) = [\text{curv}(\mathcal{G})].$$

This shows that the curvature of a bundle gerbe is a 3-form with integral class.

As a little example, we may compute the Dixmier-Douady class of a trivial bundle gerbe \mathcal{I}_ρ . It is easy to see that – for any open cover – the functions

g_{ijk} can be chosen to be constantly 1, so that the local connection 1-forms vanish, $A_{ij} = 0$. Solely the curving ρ restricts to non-trivial 2-forms $\rho|_{V_i}$ on V_i . Now it is clear that the Deligne class of \mathcal{I}_ρ is represented by $(1, 0, \rho)$, and that the Dixmier-Douady class vanishes. Indeed, the curvature is an exact form, whose class also vanishes.

Finally, let us remark that there are three standard constructions one can do with bundle gerbes: tensor products, duals and pullbacks. Without giving the details of these constructions, one observes that they behave naturally under the correspondence with classes in Deligne cohomology: the tensor product of bundle gerbes corresponds to the sum of classes, taking the dual bundle gerbe corresponds to taking opposite sign, and the pullback of bundle gerbes corresponds to the pullback of classes.

4 Morphisms of Bundle Gerbes

To find the appropriate definition of a morphism between two bundle gerbes \mathcal{G}_1 and \mathcal{G}_2 , assume for a moment that the coverings $\pi_i : Y_i \rightarrow M$ come from open covers for $i = 1, 2$. To compare the gerbe data, it would be natural to go to a common refinement of these covers. On the double intersections of this common refinement the line bundles L_1 and L_2 could be compared. For general coverings π_1 and π_2 the common refinement amounts to consider the fibre product

$$Z := Y_1 \times_M Y_2$$

thought of as a new covering $\zeta : Z \rightarrow M$ sending (y_1, y_2) to the point $\pi_1(y_1) = \pi_2(y_2)$. The two-fold intersections amount to consider $Z^{[2]}$. The restriction of the line bundle L_i to $Z^{[2]}$ is implemented by the pullback along the canonical map $y_i : Z^{[2]} \rightarrow Y_i^{[2]}$. A first idea is to require that the line bundles $y_1^*L_1$ and $y_2^*L_2$ are isomorphic. In fact, this was the original definition of a morphism between bundle gerbes [Mur96]. However, it turned out that this definition was too restrictive, in other words: the isomorphism classes were too small, and there were many non-isomorphic bundle gerbes having the same Deligne class and hence the same surface holonomy.

A solution to this was presented in [MS00]: the line bundles shouldn't be isomorphic but stably isomorphic in the sense that there is a line bundle $A \rightarrow Z$ with an isomorphism

$$y_1^*L_1 \otimes \zeta_2^*A \cong \zeta_1^*A \otimes y_2^*L_2$$

of line bundles over $Z^{[2]}$. Here ζ_1 and ζ_2 are two natural projections from $Z^{[2]}$ to Z . It is natural to demand that the data of a morphism of bundle gerbes

– the line bundle $A \rightarrow Z$ and an isomorphism α as above – is compatible with the rest of the structure of the bundle gerbes, namely the curvings and the groupoid multiplications.

Summarizing, with the additional generalization that A may be a vector bundle (of rank higher than 1), the correct definition of a morphism between bundle gerbes is

Definition 2. *A morphism $\mathcal{A} : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ of bundle gerbes is a vector bundle $A \rightarrow Z$ together with an isomorphism*

$$\alpha : y_1^* L_1 \otimes \zeta_2^* A \rightarrow \zeta_1^* A \otimes y_2^* L_2$$

of vector bundles over $Z^{[2]}$. Two axioms have to be satisfied:

(M1) *the curvature of A is a real 2-form and fixed by*

$$\text{curv}(A) = y_2^* C_2 - y_1^* C_1,$$

(M2) *the isomorphism α commutes with the groupoid multiplications μ_1 and μ_2 of the bundle gerbes in the sense that the diagram*

$$\begin{array}{ccc} y_1^* \pi_{12}^* L_1 \otimes y_1^* \pi_{23}^* L_1 \otimes \zeta_3^* A & \xrightarrow{y_1^* \mu_1 \otimes \text{id}} & y_1^* \pi_{13}^* L_1 \otimes \zeta_3^* A \\ \downarrow 1 \otimes \zeta_{23}^* \alpha & & \downarrow \zeta_{13}^* \alpha \\ \zeta_{12}^* y_1^* L_1 \otimes \zeta_2^* A \otimes \zeta_{23}^* y_2^* L_2 & & \\ \downarrow \zeta_{12}^* \alpha \otimes 1 & & \downarrow \\ \zeta_1^* A \otimes y_2^* \pi_{12}^* L_2 \otimes y_2^* \pi_{23}^* L_2 & \xrightarrow{1 \otimes y_2^* \mu_2} & \zeta_1^* A \otimes y_2^* \pi_{13}^* L_2 \end{array}$$

of isomorphisms of vector bundles over $Z^{[3]}$ is commutative.

There is one important point to notice from this definition of a morphism. Given two such morphisms

$$\mathcal{A} : \mathcal{G}_1 \rightarrow \mathcal{G}_2 \quad \text{and} \quad \mathcal{A}' : \mathcal{G}_1 \rightarrow \mathcal{G}_2$$

each providing a vector bundle A, A' over Z , to compare both morphisms it doesn't make sense to state that they are equal or not: to compare vector bundles one needs isomorphisms between them. This leads us forthright to the fact that bundle gerbes form a 2-category [Ste00]: we have bundle gerbes as objects, morphisms between the objects as defined above, and 2-morphisms between the morphisms. Before we discuss the 2-categorical aspects of bundle gerbes, we give the precise definition of a 2-morphism.

Definition 3. Let $\mathcal{A} : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ and $\mathcal{A}' : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ be two morphisms. A 2-morphism

$$\beta : \mathcal{A} \Rightarrow \mathcal{A}'$$

is an isomorphism $\beta : A \rightarrow A'$ of vector bundles over Z , which is compatible with the isomorphisms α and α' in the sense that the diagram

$$\begin{array}{ccc} y_1^* L_1 \otimes \zeta_2^* A & \xrightarrow{\alpha} & \zeta_1^* A \otimes y_2^* L_2 \\ 1 \otimes \zeta_2^* \beta \downarrow & & \downarrow \zeta_1^* \beta \otimes 1 \\ y_1^* L_1 \otimes \zeta_2^* A' & \xrightarrow{\alpha'} & \zeta_1^* A' \otimes y_2^* L_2 \end{array}$$

of isomorphisms of vector bundles over $Z^{[2]}$ is commutative.

Realizing that bundle gerbes form a 2-category is not a fault of the theory, it is a feature. To give an example, notice that – as in every 2-category – the set of all morphisms between two fixed bundle gerbes \mathcal{G}_1 and \mathcal{G}_2 , together with the set of all the 2-morphisms between such morphisms, forms a category. To be a bit more precise: for bundle gerbes they form a groupoid, since all 2-morphisms are invertible. In particular, we have a groupoid of endomorphisms from a bundle gerbe \mathcal{G} to itself. This groupoid may be considered as the groupoid of gauge transformations of \mathcal{G} . So we get a clear understanding what the gauge symmetry of a gauge theory for strings is: it is a groupoid rather than a group.

Another feature of this 2-categorical point of view is the following. In any 2-category, a morphism $\mathcal{A} : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ is called isomorphism, if it is invertible in the sense that there is another morphism $\mathcal{B} : \mathcal{G}_2 \rightarrow \mathcal{G}_1$ in the opposite direction, such that there are 2-isomorphisms

$$\mathcal{B} \circ \mathcal{A} \Rightarrow \text{id}_{\mathcal{G}_1} \quad \text{and} \quad \mathcal{A} \circ \mathcal{B} \Rightarrow \text{id}_{\mathcal{G}_2}.$$

Now we ask, what this general definition means for morphisms of bundle gerbes from Definition 2. Of course one has to say two things: what are the identity morphisms $\text{id}_{\mathcal{G}_1}$ and $\text{id}_{\mathcal{G}_2}$, and how the composition \circ is defined. The last point is quite involved, however it can be done [Ste00]. In this article we only present the result.

Proposition 1. A morphism $\mathcal{A} : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ of bundle gerbes is an isomorphism, if and only if the vector bundle $A \rightarrow Z$ is a line bundle, i.e. has rank 1.

The standard literature about bundle gerbes, e.g. [CJM02, GR02], takes the last proposition as the definition of morphisms between bundle gerbes

(so-called stable isomorphisms), and neglects that there are morphisms with vector bundles of higher rank. As a consequence, in the standard literature bundle gerbes form a 2-groupoid rather than a 2-category. The advantages of our definition become apparent later on.

For the rest of this section, we restrict ourselves to isomorphisms between bundle gerbes. We would like to get an overview over the set of possible isomorphisms between \mathcal{G}_1 and \mathcal{G}_2 . First we get rid of the 2-morphisms by going to equivalence classes: we call two such isomorphisms equivalent, if there is a 2-morphism between them (which is automatically a 2-isomorphism). In other words, we consider the skeleton of the groupoid of isomorphisms between \mathcal{G}_1 and \mathcal{G}_2 .

Proposition 2 ([CJM02, SSW05]). *The set of equivalence classes of isomorphisms between two fixed bundle gerbes is a torsor over the group $\text{Pic}_0(M)$ of isomorphism classes of flat line bundles over M .*

Supposed we are able to compose morphisms as indicated above, being isomorphic is an equivalence relation on the set of bundle gerbes. We present a result which classifies these isomorphism classes of bundle gerbes, in other words: we are looking for the skeleton of the 2-category of bundle gerbes. Recall that a bundle gerbe defines by its local data a cocycle in the Deligne cochain group $\mathcal{D}^2(2)$. Similarly it can be shown, that an isomorphism $\mathcal{A} : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ defines a cochain (t, W) in the cochain group $\mathcal{D}^1(2)$. It relates the cocycles of the two bundle gerbes by its coboundary,

$$(g_2, A_2, B_2) = (g_1, A_1, B_1) + D(t, W).$$

This equation means that two isomorphic bundle gerbes define the same class in Deligne cohomology. Even stronger is the following theorem.

Theorem 1 ([MS00]). *The set of isomorphism classes of bundle gerbes is in bijection to the Deligne cohomology group $H^2(M, \mathcal{D}(2))$.*

This theorem makes contact to one of the other definitions of gerbes with connective structure and gives the precise relation between line bundles and bundle gerbes: line bundles are classified by the Deligne cohomology group $H^1(M, \mathcal{D}(1))$, and bundle gerbes by $H^2(M, \mathcal{D}(2))$.

In the remainder of this section, we are going to sketch the relation between bundle gerbes and yet another realization of gerbes, namely sheaves of groupoids. To this purpose we need the trivial bundle gerbes \mathcal{I}_ρ defined in the last section as an example for bundle gerbes. In the same way as a trivialization of a line bundle is an isomorphism to a trivial line bundle, we say

Definition 4. A *trivialization of a bundle gerbe* \mathcal{G} is an isomorphism

$$\mathcal{T} : \mathcal{G} \rightarrow \mathcal{I}_\rho.$$

Let us briefly exhibit the details of a trivialization, which follow from the definitions of an isomorphism and of the trivial bundle gerbe \mathcal{I}_ρ . The isomorphism \mathcal{T} consists of a line bundle T over the space $Z = Y \times_M M$ which we identify canonically with Y itself. Under this identification, the two projections to the coverings of the bundle gerbes become the identity $\text{id} : Y \rightarrow Y$ and the covering $\pi : Y \rightarrow M$ itself, so that axiom (M1) becomes

$$\text{curv}(T) = \pi^* \rho - C.$$

Further, the isomorphism \mathcal{T} consists of an isomorphism $\tau : L \otimes \pi_2^* T \rightarrow \pi_1^* T$ of line bundles over $Z^{[2]} = Y^{[2]}$. Because the groupoid multiplication of the trivial bundle gerbe \mathcal{I}_ρ is the identity, axiom (M2) for τ reduces to the equation

$$\pi_{13}^* \tau \circ \mu = \pi_{12}^* \tau \circ \pi_{23}^* \tau$$

of isomorphisms of line bundles over $Z^{[3]} = Y^{[3]}$.

Of course not every bundle gerbe admits a trivialization. In the same way as for line bundles the obstruction to the existence of a trivialization is given by the first Chern class, a bundle gerbe \mathcal{G} admits a trivialization if and only if its Dixmier-Douady class vanishes [CJM02]. In this case, the curvature of the bundle gerbe \mathcal{G} is an exact form, and

$$\text{curv}(\mathcal{G}) = d\rho$$

for any trivialization $\mathcal{T} : \mathcal{G} \rightarrow \mathcal{I}_\rho$.

We define a sheaf of groupoids in the following way: for an open subset U of M consider the set of trivializations $\mathcal{G}|_U \rightarrow \mathcal{I}_\rho$ for all 2-forms ρ (the set may be empty). Since trivializations are nothing but isomorphisms of bundle gerbes, together with the 2-morphisms they form naturally a groupoid $\mathcal{G}(U)$. Furthermore, trivializations can clearly be restricted to smaller subsets, so that the assignment

$$U \mapsto \mathcal{G}(U)$$

is a presheaf of groupoids. The gluing axiom can be shown by gluing together two trivializations over U_1 and U_2 , if there is a 2-morphisms between their restrictions to the intersection $U_1 \cap U_2$. This way, every bundle gerbe defines a sheaf of groupoids in Brylinski's sense [MS00].

5 Holonomy around closed oriented Surfaces

The holonomy of a bundle gerbe around a closed oriented surface should be analogous to the holonomy of a line bundle around a loop $\gamma : S^1 \rightarrow M$, since S^1 is also closed and oriented.

So, it is worthwhile to recall how the holonomy of a line bundle L over M around a loop $\gamma : S^1 \rightarrow M$ can be defined. The pullback of L along γ gives a line bundle over the circle, whose first Chern class vanishes for dimensional reasons. Hence, it becomes isomorphic to a trivial line bundle with some connection 1-form $A \in \Omega^1(S^1)$. Then,

$$\text{hol}_L(\gamma) := \exp \left(\int_{S^1} A \right)$$

is a number in $U(1)$ which is in fact independent of the choice of the trivialization. We also write out this definition in terms of local data (g, A) of the line bundle L with respect to an open cover $\{V_i\}_{i \in I}$. Choose a triangulation Δ of S^1 that is subordinated to the open cover by a map $i : \Delta \rightarrow I$, such that $\gamma(e) \subset V_{i(e)}$ for any edge e and $\gamma(v) \in V_{i(v)}$ for any vertex v . Then, by splitting the integral over A with respect to the triangulation and using Stokes' Theorem, one can derive the formula

$$\text{hol}_L(\gamma) := \prod_{i \in I} \exp \left(\int_e \gamma^* A_{i(e)} \right) \cdot \prod_{v \in \partial e} g_{i(e)i(v)}^{\epsilon(e,v)}(\gamma(v))$$

where $\epsilon(e, v) \in \{-1, 1\}$ is positive, if v is the endpoint of e and negative otherwise. The meaning of this formula is that one has to integrate the local connection 1-forms along the edges, and to use the transition functions to intermediate at the vertices between two edges.

For the definition of the holonomy of a bundle gerbe \mathcal{G} we start with a configuration like in Figure 1 and mimic the same procedure as for line bundles.

Definition 5 ([CJM02]). *Let \mathcal{G} be a bundle gerbe over M . For a closed oriented surface Σ and a smooth map $\phi : \Sigma \rightarrow M$, let*

$$\mathcal{T} : \phi^* \mathcal{G} \rightarrow \mathcal{I}_\rho$$

be a trivialization of the pullback of the bundle gerbe \mathcal{G} along ϕ . Then we define

$$\text{hol}_{\mathcal{G}}(\phi) := \exp \left(\int_{\Sigma} \rho \right)$$

to be the holonomy of the bundle gerbe \mathcal{G} around $\phi : \Sigma \rightarrow M$.

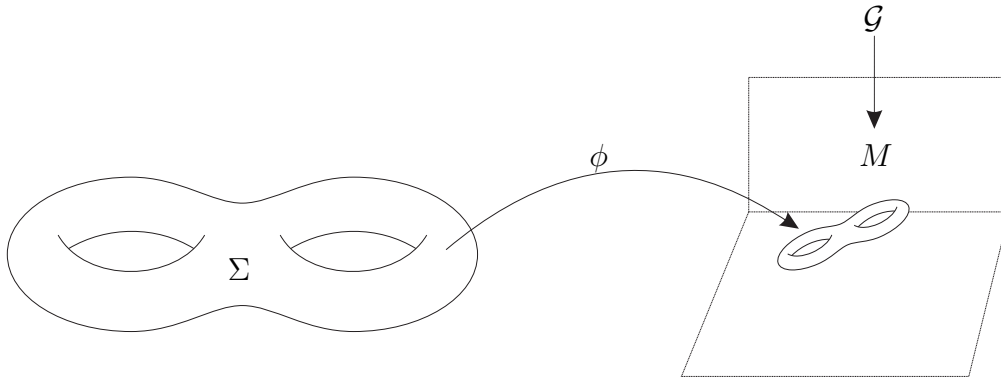


Figure 1: A surface is mapped into some space M with bundle gerbe \mathcal{G}

This sounds easy but we have to assure that the number $\text{hol}_{\mathcal{G}}(\phi)$ is independent of the choice of the trivialization, which exists for dimensional reasons. Different trivializations may have different 2-forms ρ , however, in the next lemma we show that the difference $\rho_2 - \rho_1$ between two such 2-forms is the curvature of some line bundle over M , in particular: it is a closed form with integral class. Then, the calculation

$$\exp\left(\int_{\Sigma} \rho_2\right) = \exp\left(\int_{\Sigma} \rho_2 - \rho_1\right) \cdot \exp\left(\int_{\Sigma} \rho_1\right) = \exp\left(\int_{\Sigma} \rho_1\right)$$

shows that the definition $\text{hol}_{\mathcal{G}}(\phi)$ is independent of the choice of the trivialization.

Lemma 1. *Two trivializations*

$$\mathcal{T}_1 : \mathcal{G} \rightarrow \mathcal{I}_{\rho_1} \quad \text{and} \quad \mathcal{T}_2 : \mathcal{G} \rightarrow \mathcal{I}_{\rho_2}$$

of the same bundle gerbe \mathcal{G} over M determine a line bundle over M with curvature $\rho_2 - \rho_1$.

Proof. Using the features of the 2-category of bundle gerbes, we can give a very relaxed proof: by taking the inverse and composition, we obtain an isomorphism

$$\mathcal{T}_2 \circ \mathcal{T}_1^{-1} : \mathcal{I}_{\rho_1} \rightarrow \mathcal{I}_{\rho_2}$$

of trivial bundle gerbes. From the definitions of isomorphisms and trivial bundle gerbes it follows immediately, that $\mathcal{T}_2 \circ \mathcal{T}_1^{-1}$ is a line bundle with

curvature $\rho_2 - \rho_1$. But since we have not defined inverses and composition of isomorphisms, let us also give a more concrete proof. Recall that the two trivializations provide line bundles T_1 and T_2 over Y . We show that the line bundle $T_2 \otimes T_1^*$ descends along $\pi : Y \rightarrow M$. To do so we have to specify descent data, namely an isomorphism

$$\chi : \pi_2^*(T_2 \otimes T_1^*) \rightarrow \pi_1^*(T_1 \otimes T_2^*)$$

of line bundles over $Y^{[2]}$ which satisfies the cocycle condition

$$\pi_{13}^*\chi = \pi_{12}^*\chi \circ \pi_{23}^*\chi$$

over $Y^{[3]}$. For the definition of χ recall that the two trivializations provide isomorphisms $\tau_i : L \otimes \pi_2^*T_i \rightarrow \pi_1^*T_i$ for $i = 1, 2$. Then we declare χ to be the following isomorphism:

$$\begin{aligned} \chi : \pi_2^*(T_2 \otimes T_1^*) &\cong (L \otimes \pi_2^*T_2) \otimes (\pi_2^*T_1^* \otimes L^*) \\ &\downarrow \tau_1 \otimes \tau_2^{-1} \\ \pi_1^*T_2 \otimes \pi_1^*T_1^* &\cong \pi_1^*(T_2 \otimes T_1^*) \end{aligned}$$

Using axiom (M2) for τ_1 and τ_2 one can now show that χ satisfies the cocycle condition. Hence, $T_2 \otimes T_1^*$ descends to a line bundle N over M with the property

$$\pi^*N \cong T_2 \otimes T_1^*.$$

To finish the proof we have to compute the curvature of N . Notice that by axiom (M1) we find

$$\text{curv}(T_2 \otimes T_1^*) = \pi^*\rho_2 - C - (\pi^*\rho_1 - C) = \pi^*(\rho_2 - \rho_1).$$

This shows that the curvature of N is $\rho_2 - \rho_1$. □

Now that we have the definition of holonomy around a closed oriented surface we address the question if it provides the desired generalization of Stokes' Theorem, which is important for the application in Wess-Zumino-Witten models.

Proposition 3. *Let \mathcal{G} be a bundle gerbe over M with curvature H . For a three-dimensional oriented manifold B with boundary and a map $\Phi : B \rightarrow M$, we find*

$$\text{hol}_{\mathcal{G}}(\Phi|_{\partial B}) = \exp \left(\int_B \Phi^*H \right).$$

Proof. Remember that for any trivialization $\mathcal{T} : \Phi^*\mathcal{G}|_{\partial B} \rightarrow \mathcal{I}_\rho$ we have $\Phi^*H|_{\partial B} = d\rho$. Then, by definition,

$$\text{hol}_{\mathcal{G}}(\Phi|_{\partial B}) = \exp\left(\int_{\partial B} \rho\right) = \exp\left(\int_B \Phi^*H\right)$$

It is important to recognize that the last step does not just follow from Stokes' Theorem (because Φ^*H is not exact over B). In fact one can triangulate B , choose local trivializations with 2-forms ρ_i differing by closed 2-forms with integral class by Lemma 1, and then use Stokes' Theorem on each 3-face. The remaining differences between the 2-forms disappear after exponentiation. \square

This way we reproduce the amplitude of the coupling term of the Wess-Zumino-Witten model by

$$\mathcal{A}(\phi) = \exp(S_{\text{kin}}(\phi)) \cdot \text{hol}_{\mathcal{G}}(\phi).$$

Notice that we did not impose any condition on the topology of the target space M . While in Witten's definition, the background field (apart from the metric) is just the three 3-form H , in the approach using bundle gerbes the background field is the bundle gerbe \mathcal{G} . This is no contradiction since that all bundle gerbes over $SU(2)$ with curvature $\text{curv}(\mathcal{G}) = H$ are isomorphic [GR02]. However, for general target spaces there may be bundle gerbes with same curvature, which are not isomorphic. This occurs for instance on the Lie group $Spin(4n)/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. Then, the bundle gerbe contains more information as just its curvature, and it becomes essential to recognize the bundle gerbe itself as the background field which defines the theory. Just as little as by its curvature, the bundle gerbe can be replaced by a 2-form, sometimes called the B-field, or Kalb-Ramond field. Such a 2-form ρ exists in general only locally, namely when there is a trivialization $\mathcal{G}|_U \rightarrow \mathcal{I}_\rho$. Even then, it is not unique.

This situation is analogous in every detail to what is called the Aharonov-Bohm effect in electrodynamics: in a (first) quantized theory of charged particles moving through a certain electric field of field strength F , not only the field strength, but also the gauge potential A is an observable quantity which is necessary to describe the theory – opposed to a classical theory of electrodynamics where everything is determined by the field strength and Maxwell's equations. This Aharonov-Bohm effect (which in fact was predicted 10 years before by W. Ehrenberg and R.E. Siday) has been measured. Now one might think that the gauge potential A , a 1-form, is the object which describes the theory. But the gauge potential with $dA = F$ is only defined locally, and if it

is, it is not unique. This is exactly the behaviour of a line bundle and shows that a line bundle provides the correct description for this situation.

To close this section, we reformulate the holonomy in terms of local data of the bundle gerbe, analogous the formula for the holonomy of a line bundle. Recall that a trivialization

$$\mathcal{T} : \phi^* \mathcal{G} \rightarrow \mathcal{I}_\rho$$

chosen in Definition 5 implies the following relation between the local data of the bundle gerbe and the local data of the trivial gerbe,

$$(1, 0, \rho) = \phi^*(g, A, B) + D(t, W).$$

Now, following the strategy of the local expression for the line bundle, we choose a triangulation Δ of the surface Σ , consisting of faces f , edges e and vertices v . It should be chosen subordinated to the same open cover $\{V_i\}_{i \in I}$ of M which was used to extract the local data (g, A, B) of the bundle gerbe \mathcal{G} . So there is a map $i : \Delta \rightarrow I$, assigning to each face, edge or vertex f an index $i(f)$ so that $\phi(f) \subset V_{i(f)}$. Now the integral of the 2-form ρ over Σ which defines the holonomy may be split up with respect to the triangulation. By a subsequent use of Stokes' Theorem and the above formula for the local data [CJM02], one ends up with the following formula

$$\begin{aligned} \text{hol}_{\mathcal{G}}(\phi) = & \prod_{f \in \Delta} \exp \left(\int_f \phi^* B_{i(f)} \right) \\ & \cdot \prod_{e \in \partial f} \exp \left(\int_e \phi^* A_{i(f), i(e)} \right) \cdot \prod_{v \in \partial e} g_{i(f), i(e), i(v)}^{\epsilon(f, e, v)}(\phi(v)). \quad (2) \end{aligned}$$

This formula shows explicitly what's going on: the surface integral is expressed by local integrals over the local 2-forms B_i , the local Kalb-Ramond fields with $dB_i = H$. But on the edges and vertices, the sum has to be corrected by the rest of the local data of the bundle gerbe. This way, the triangulated surface gets decorated like shown in Figure 2.

Of course one can define the last expression without knowing bundle gerbes just by starting with a class in Deligne cohomology represented by local data (g, A, B) . In fact, surface holonomy appeared first in this form and was recognized to be useful for string theory [Alv85, Gaw88].

6 The Line Bundle over the Loop Space

As pointed out in the introduction, the holonomy of a line bundle L over a manifold M can be seen as a $U(1)$ -valued function

$$\text{hol}_L : LM \rightarrow U(1)$$

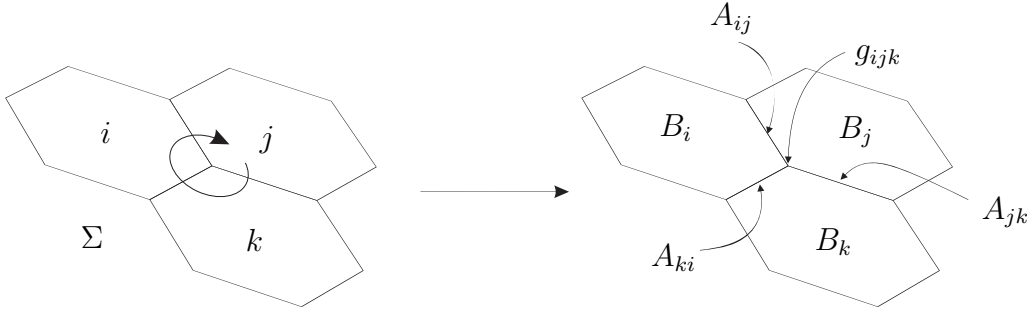


Figure 2: The triangulation of the surface Σ is decorated by the local data: the faces with 2-forms B_i , the edges with 1-forms A_{ij} and the vertices with the functions g_{ijk} .

on its loop space. Recall that such a function is the first object in a series of mathematical objects and is followed by a line bundle. So one might guess that, when starting with a bundle gerbe instead of a line bundle, the function on the loop space is replaced by a line bundle over the loop space. This correspondence in question between gerbes over M and line bundles over LM was first discovered in terms of Deligne cohomology [Gaw88]. It was later redefined and extended to sheaves of groupoids [Bry93]. The associated homomorphism of Deligne cohomology groups

$$H^k(M, \mathcal{D}(k)) \rightarrow H^{k-1}(LM, \mathcal{D}(k-1))$$

is called transgression, see also [GT01]. In particular, it reproduces for $k = 1$ the correspondence between line bundles over M and $U(1)$ -valued functions on LM .

Let us describe briefly how one can construct the line bundle from a bundle gerbe \mathcal{G} over M . The construction we present here is an adaption of Brylinski's construction to bundle gerbes. Another construction starting from bundle gerbes is proposed in [GR02]. The fibre over a loop $\gamma : S^1 \rightarrow M$ consists of all trivializations

$$\mathcal{T} : \gamma^* \mathcal{G} \rightarrow \mathcal{I}_0.$$

Such trivializations exist, because the Dixmier-Douady class of the pullback bundle gerbe lives in $H^3(S^1, \mathbb{Z}) = 0$. We identify two trivializations \mathcal{T}_1 and \mathcal{T}_2 , if there is a 2-morphism

$$\mathcal{T}_1 \Rightarrow \mathcal{T}_2.$$

After that, the fibre over γ consists of equivalence classes of isomorphisms between $\gamma^*\mathcal{G}$ and \mathcal{I}_0 . This set is by Proposition 2 a torsor over the group $\text{Pic}_0(S^1)$ of isomorphism classes of flat line bundles over the circle. It is well-known that this group is canonically isomorphic to $U(1)$. Under this identification, the fibre over each loop γ is in a natural way a $U(1)$ -torsor. One can further show that the union of the fibres carries a canonical smooth structure, which makes it into a principal $U(1)$ -bundle over LM .

On the associated line bundle L the bundle gerbe defines a connection. The Deligne cohomology class of this line bundle (with connection) is the image of the class of the bundle gerbe \mathcal{G} we started with under the transgression homomorphism of Deligne cohomology groups. There is an interesting relation between the holonomy of the line bundle L over LM and the holonomy of the bundle gerbe \mathcal{G} over M : if $\gamma : S^1 \rightarrow LM$ is a loop in the loop space we can naturally identify it with a map $\phi : S^1 \times S^1 \rightarrow M$. One can now consider both the holonomy of the line bundle L around the loop γ as well as the holonomy of the bundle gerbe \mathcal{G} around ϕ . Both coincide [Bry93]:

$$\text{hol}_L(\gamma) = \text{hol}_{\mathcal{G}}(\phi).$$

The line bundle L over the loop space plays an important role for Wess-Zumino-Witten models on Lie groups G . Its total space can be endowed with a group structure in a way that it becomes a central extension of the loop group LG . It can also be completed with respect to an appropriate scalar product, so that the space of holomorphic sections forms a Hilbert space, which acts as the space of states for the quantized Wess-Zumino-Witten model [Gaw99].

7 D-Branes and Surfaces with Boundary

Some string theories involve not only loops (closed strings) but also open strings, whose worldsheets are surfaces with boundary. For those, we are not able to apply Definition 5 of the holonomy of a bundle gerbe \mathcal{G} : the integral of the closed form $\rho_2 - \rho_1$ with integral class isn't anymore an integer. More precisely, a boundary term emerges which has to be compensated to achieve a holonomy independent of the choice of the trivialization.

Unfortunately, there is no analogous situation for the holonomy of line bundles. Therefore, we adopt the concept of D-branes from string theory [Pol96]. D-Branes restrict the endpoints of open strings to submanifolds Q of the target space, and hence impose (generalized) Dirichlet boundary conditions to the motion of open strings. In the former definition of the Wess-Zumino-Witten model on a Lie group G by a 3-form H , a typical choice of the

submanifold Q is a (twisted) conjugacy class of G . Here there is an additional condition, namely that the 3-form is fixed on the D-brane to $H|_Q = d\omega$ for some 2-form ω on Q . This 2-form can be interpreted as the field strength of a twisted $U(1)$ -gauge field on Q . Let us for simplicity assume that the surface Σ has only one boundary component. Accordingly, there is one D-brane (Q, ω) chosen, and we consider a map $\phi : \Sigma \rightarrow G$ with

$$\phi(\partial\Sigma) \subset Q.$$

Then, the following definition of the amplitudes is given [Gaw99]. Let D^2 be a disk acting as a cap for the surface Σ , so that there exists a three-dimensional manifold B whose boundary is $\partial B = \Sigma \cup D^2$. We again have to assume that there is an extension $\Phi : B \rightarrow M$ of the map ϕ , which now has to send the cap D^2 into the D-brane Q . Then, the amplitude is defined by

$$\mathcal{A}(\phi) := \exp \left(S_{\text{kin}}(\phi) + \int_B \Phi^* H - \int_{D^2} \Phi^* \omega \right).$$

There is a condition on the well-definedness of this amplitude; here it is not sufficient that H is a closed 3-form with integral class. The equations $dH = 0$ and $H|_Q = d\omega$ mean that the pair (H, ω) defines a class in the relative cohomology $H^3(G, Q, \mathbb{R})$, and the condition is, that this class is integral in the sense that it lies in the image of the relative cohomology with integer coefficients. There are explicit expressions for ω in the case that Q is a (twisted) conjugacy class, so that this integrality condition is satisfied.

We have learned before that the theory of bundle gerbes extends the former 3-form approach to the Wess-Zumino-Witten-model. Accordingly, we have to adjust the definition of a D-Brane. It is still build up on a submanifold Q . In the first attempt [GR02], the 2-form ω was replaced by a trivialization

$$\mathcal{E} : \mathcal{G}|_Q \rightarrow \mathcal{I}_\omega$$

of the bundle gerbe \mathcal{G} restricted to Q . Notice that this reproduces in particular the old condition $H|_Q = d\omega$ for the curvature H of the bundle gerbe \mathcal{G} . Later it was recognized [Gaw05] that having a trivialization, i.e. an isomorphism, was too strong. In fact a D-brane for a certain bundle gerbe over $SO = (4n)/\mathbb{Z}_2$ was found which doesn't admit an isomorphism but a weaker structure – a morphism from $\mathcal{G}|_Q$ to the trivial bundle gerbe \mathcal{I}_ω .

Such a morphism $\mathcal{G} \rightarrow \mathcal{I}_\omega$ is also called a \mathcal{G} -module or bundle gerbe module. A slightly more general version was also considered in a purely mathematical context to obtain a geometric realization of twisted K-theory [BCM⁺02].

Definition 6. Let \mathcal{G} be a bundle gerbe over M . A D -brane for \mathcal{G} is a submanifold Q of M together with a $\mathcal{G}|_Q$ -module

$$\mathcal{E} : \mathcal{G}|_Q \rightarrow \mathcal{I}_\omega.$$

The 2-form ω on Q is called the curvature of the D -brane.

Recall that Lemma 1 implies that two trivialisations $\mathcal{G} \rightarrow \mathcal{I}_{\rho_1}$ and $\mathcal{G} \rightarrow \mathcal{I}_{\rho_2}$ lead to a line bundle over M of curvature $\rho_2 - \rho_1$. The same statement holds for a gerbe module $\mathcal{G} \rightarrow \mathcal{I}_\omega$ and an trivialisaton $\mathcal{G} \rightarrow \mathcal{I}_\rho$: they define a vector bundle $E \rightarrow M$ of curvature $\omega - \rho$. To prove this, recall that a gerbe module is a trivialisaton with a vector bundle instead of a line bundle. To this situation, the proof of Lemma 1 extends without changes. The holonomy of this vector bundle will be the term which compensates the changes of ρ on the boundary.

Now consider a configuration like shown in Figure 3. We give the following definition of holonomy.

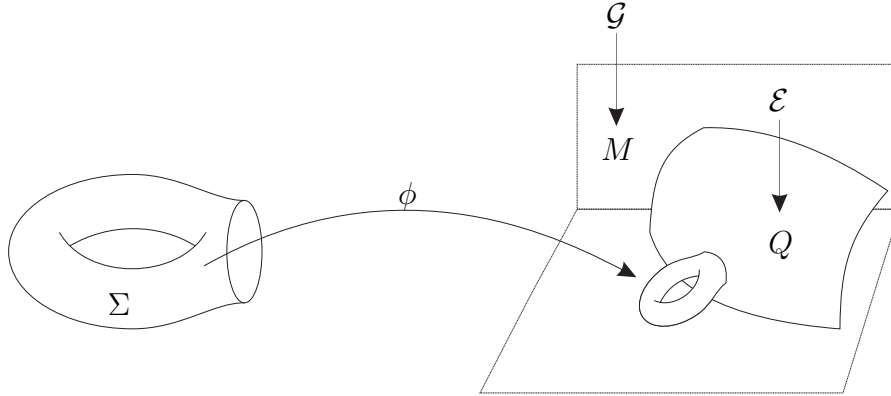


Figure 3: A surface is mapped into a target space with bundle gerbe \mathcal{G} , so that its boundary is mapped into the submanifold Q with bundle gerbe module \mathcal{E} .

Definition 7 ([CJM02]). Let \mathcal{G} be a bundle gerbe over M , and let (Q, \mathcal{E}) be a D -brane for \mathcal{G} . For an oriented surface Σ and a map $\phi : \Sigma \rightarrow M$ which maps the boundary of Σ into Q , choose a trivialisaton

$$\mathcal{T} : \phi^* \mathcal{G} \rightarrow \mathcal{I}_\rho$$

of the pullback of \mathcal{G} along ϕ . Its restriction to $\partial\Sigma$ determines in combination with the bundle gerbe module

$$\phi^*\mathcal{E} : \phi^*\mathcal{G}|_{\partial\Sigma} \rightarrow \mathcal{I}_{\phi^*\omega}$$

a vector bundle $E \rightarrow \partial\Sigma$ of curvature $\phi^*\omega - \rho$. Then, we define

$$\text{hol}_{\mathcal{G},\mathcal{E}}(\phi) = \exp\left(\int_{\Sigma} \rho\right) \cdot \text{tr}(\text{hol}_E(\partial\Sigma))$$

to be the holonomy of the bundle gerbe \mathcal{G} with D-Brane (Q, \mathcal{E}) around ϕ .

In fact it is easy to see that this definition does not depend on the choice of the trivialization \mathcal{T} : for another trivialization \mathcal{T}' recall that by Lemma 1 we obtain a line bundle $N \rightarrow \partial\Sigma$ with curvature $\rho' - \rho$. The following change in the first factor of $\text{hol}_{\mathcal{G},\mathcal{E}}(\phi)$ emerges:

$$\begin{aligned} \exp\left(\int_{\Sigma} \rho\right) &= \exp\left(\int_{\Sigma} \rho'\right) \cdot \exp\left(\int_{\Sigma} -\text{curv}(N)\right) \\ &= \exp\left(\int_{\Sigma} \rho'\right) \cdot (\text{hol}_N(\partial\Sigma))^{-1}. \end{aligned}$$

This change has to be compensated by the second factor. Indeed, the second trivialization determines another vector bundle $E' \rightarrow \partial\Sigma$ of curvature $\phi^*\omega - \rho'$. From the construction of these bundles in the proof of Lemma 1 it becomes clear that they satisfy

$$E \cong N \otimes E'.$$

This means for the second factor

$$\text{tr}(\text{hol}_E(\partial\Sigma)) = \text{tr}(\text{hol}_{E' \otimes N}(\partial\Sigma)) = \text{hol}_N(\partial\Sigma) \cdot \text{tr}(\text{hol}_{E'}(\partial\Sigma))$$

Thus we have shown that the holonomy defined in Definition 7 does not depend on the choice of the trivialization.

With this definition of holonomy around a surface with boundary, we have to check that it reproduces the amplitude given above in terms of the 3-form H on M and the 2-form ω on Q .

Proposition 4. *Let \mathcal{G} be a bundle gerbe over M with curvature H and let (Q, \mathcal{E}) be a D-brane with curvature ω . For an oriented three-dimensional manifold B , whose boundary decomposes in two parts Σ and D^2 , and a map $\Phi : B \rightarrow M$ with $\Phi(D^2) \subset Q$ we find*

$$\text{hol}_{\mathcal{G},\mathcal{E}}(\Phi|_{\Sigma}) = \exp\left(\int_B \Phi^*H - \int_{D^2} \Phi^*\omega\right).$$

This proposition can be proven similarly to Proposition 3. It shows that the theory of bundle gerbes and bundle gerbe modules is helpful for open string theories and allows for a proper definition of the amplitudes of world-sheets with boundary in the Wess-Zumino-Witten model – even in topologically non-trivial situations.

As a last point, we have a look on the holonomy in terms of local data. Let (g, A, B) be local data of the bundle gerbe \mathcal{G} with respect to an open cover $\{V_i\}_{i \in I}$ of M . Similarly to the local data of an isomorphism it is possible to extract local data of a morphism, in particular of a bundle gerbe module [Gaw05]. It consists of functions

$$G_{ij} : V_i \cap V_j \rightarrow U(n)$$

on double intersections, where n is the rank of the vector bundle, which is part of the structure of a morphism. It consists further of $\mathfrak{u}(n)$ -valued 1-forms P_i on each open set V_i . Like an isomorphism, a morphism

$$\mathcal{E} : \mathcal{G} \rightarrow \mathcal{I}_\omega$$

relates the local data (g, A, B) of the bundle gerbe \mathcal{G} to that of the trivial bundle gerbe \mathcal{I}_ω by the following equations:

$$\begin{aligned} G_{ij} \cdot G_{jk} \cdot G_{ik}^{-1} \cdot g_{ijk} &= 1 \\ P_j - \text{Ad}_{G_{ij}}(P_i) - G_{ij}^{-1} dG_{ij} + A_{ij} &= 0 \\ dP_i + B_i &= \omega \end{aligned}$$

In these equations, we identify $U(1)$ with the diagonal subgroup of $U(n)$, and correspondingly the Lie algebra of $U(1) - \mathbb{R}$ – with a subalgebra of $\mathfrak{u}(n)$. Notice that if the local data of the bundle gerbe is trivial, i.e. $(g, A, B) = (1, 0, 0)$, the three equations would be the usual cocycle conditions for a $U(n)$ -vector bundle with connection of curvature ω . With non-trivial local data, these cocycle conditions become twisted – for this reason, gerbe modules are also known as twisted vector bundles.

We use a triangulation of Σ which is subordinated to the open cover of M just like we did to derive the formula (2) of the local expression of the holonomy around a closed surface. Splitting of the integral of the 2-form ρ over Σ , which build the first factor of $\text{hol}_{\mathcal{G}, \mathcal{E}}(\phi)$, leads exactly to formula (2). It has to be amended by the local expression for the second factor, which is the holonomy of the vector bundle E around the boundary of Σ . The ladder is similar to the local expression for the holonomy of a line bundle around a loop, namely

$$\text{tr}(\text{hol}_E(\partial\Sigma)) = \text{tr} \text{P} \left\{ \prod_{e \in \Delta \cap \partial\Sigma} \exp \left(\int_e \phi^* P_{i(e)} \right) \cdot \prod_{v \in \partial e} G_{i(e), i(v)}^{\epsilon(e, v)} \right\}.$$

The only difference is that the terms now live in the non-abelian group $U(n)$ and have to be ordered with respect to the induced orientation on $\partial\Sigma$, which is indicated by the path-ordering operator P . The cyclic property of the trace assures that it does not depend on a specific point from where one starts multiplying terms. The complete picture where the local data is used is shown in Figure 4.

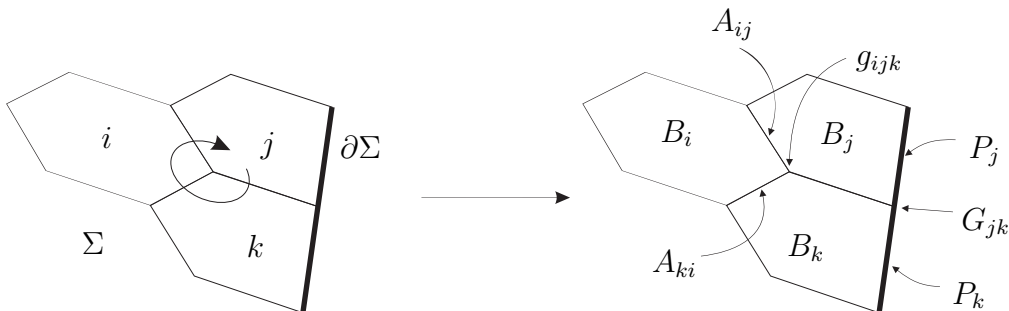


Figure 4: The triangulation of the surface Σ is decorated by the local data as in Figure 2, completed by the local 1-forms P_i and the functions G_{ij} coming from the bundle gerbe module, which are placed on the boundary $\partial\Sigma$.

8 Unoriented closed Surfaces

From the preceding section we learn two things. To incorporate boundaries we first had to choose structure – the D-branes – additional to the given bundle gerbe over M . Secondly, we restricted the possible maps $\phi : \Sigma \rightarrow M$ to those which respect this additional structure.

To incorporate unoriented surfaces Σ we also have to do these two steps. The additional structure has been defined in [SSW05] and called a Jandl structure on the bundle gerbe \mathcal{G} . It consists of an involution of M – i.e. a diffeomorphism $k : M \rightarrow M$ with $k \circ k = \text{id}_M$ – and of a certain isomorphism between the pullback bundle gerbe $k^*\mathcal{G}$ and the dual bundle gerbe \mathcal{G}^* . In the second step, we have to specify the space of maps we want to consider. As we will see, they have to be compatible in a certain sense with the involution k .

For any (unoriented) closed surface Σ there is an oriented two-fold covering $\text{pr} : \hat{\Sigma} \rightarrow \Sigma$. It is unique up to orientation-preserving diffeomorphisms

and it is connected if and only if Σ is not orientable. It has a canonical, orientation-reversing involution σ , which permutes the sheets and preserves the fibres. We call this two-fold covering the orientation covering of Σ .

Given a closed surface Σ , we consider maps $\hat{\phi} : \hat{\Sigma} \rightarrow M$ starting from the orientation covering $\hat{\Sigma}$, which are equivariant with respect to the two involutions on $\hat{\Sigma}$ and M , i.e. the diagram

$$\begin{array}{ccc} \hat{\Sigma} & \xrightarrow{\hat{\phi}} & M \\ \sigma \downarrow & & \downarrow k \\ \hat{\Sigma} & \xrightarrow{\hat{\phi}} & M \end{array}$$

has to be commutative.

Before we come to the details of a Jandl structure, let us briefly describe the idea behind its definition. If we pullback the bundle gerbe \mathcal{G} along an equivariant map $\hat{\phi}$, we obtain a bundle gerbe on the orientation covering $\hat{\Sigma}$, and could in principle compute the holonomy of this bundle gerbe around $\hat{\Sigma}$. If we do so, we would get the square of what we originally wanted, since each point of Σ is twice covered. To reveal this, we are going to establish a descent procedure – not for the bundle gerbe but rather for its holonomy. In a first attempt we assume that there is an isomorphism

$$\mathcal{A} : k^*\mathcal{G} \rightarrow \mathcal{G}^*.$$

Due to the equivariance of $\hat{\phi}$, it induces an isomorphism between $\sigma^*\hat{\phi}^*\mathcal{G}$ and $\hat{\phi}^*\mathcal{G}^*$. This isomorphism says: changing the sheet by σ goes hand in hand with replacing the bundle gerbe with its dual. With the precise definition of the dual bundle gerbe \mathcal{G}^* it becomes easy to see from Definition 7 that both processes – changing the orientation and taking the dual – give each a sign in the holonomy. So the isomorphism \mathcal{A} implies that the holonomy of $\hat{\phi}^*\mathcal{G}$ takes locally the same value on both sheets, which is the initial condition for a descent from $\hat{\Sigma}$ to Σ .

A detailed calculation shows that it is not enough to choose any isomorphism like above as additional structure. It shows that the isomorphism \mathcal{A} itself has to be equivariant in a certain sense. To give a complete definition of the Jandl structure it is convenient to use the 2-categorical language.

Definition 8. *A Jandl structure \mathcal{J} on a bundle gerbe \mathcal{G} over M is an involution k of M together with an isomorphism*

$$\mathcal{A} : k^*\mathcal{G} \rightarrow \mathcal{G}^*$$

and a 2-morphism

$$\varphi : k^* \mathcal{A} \rightrightarrows \mathcal{A}^*$$

which satisfies the equivariance condition

$$k^* \varphi = \varphi^*.$$

Notice the remarkable symmetry of the three lines. Of course we haven't developed the full theory of pullbacks and duals of morphisms and 2-morphisms in this article (although they turn out to be quite canonical constructions). For instance, $k^* \mathcal{A}$ as well as \mathcal{A}^* are both isomorphisms from \mathcal{G} to $k^* \mathcal{G}^*$, so it makes sense to have a 2-morphism φ between them. Similarly, both $k^* \varphi$ and φ^* are 2-morphisms from \mathcal{A} to $k^* \mathcal{A}^*$, so it makes sense to demand that they are equal.

To give an impression of the details of a Jandl structure, recall that an isomorphism such as \mathcal{A} consists of a line bundle A over the space Z which is build up from the two coverings of the bundle gerbes $k^* \mathcal{G}$ and \mathcal{G}^* . In this particular situation, there is a canonical lift \tilde{k} of the involution k into the space Z , and it is in fact easy to work out that the 2-morphism φ defines a \tilde{k} -equivariant structure on the line bundle A . Summarizing, a Jandl structure \mathcal{J} on \mathcal{G} is an isomorphism

$$\mathcal{A} : k^* \mathcal{G} \rightarrow \mathcal{G}^*$$

whose line bundle A is equivariant with respect to the involution \tilde{k} on Z .

As always when there is an additional structure to choose, one would like to know how many inequivalent choices there are. To say what equivalent Jandl structures are amounts to define a morphism between two of them.

Definition 9. A morphism $\beta : \mathcal{J} \rightarrow \mathcal{J}'$ between Jandl structures $\mathcal{J} = (k, \mathcal{A}, \varphi)$ and $\mathcal{J}' = (k, \mathcal{A}', \varphi')$ on the same bundle gerbe \mathcal{G} over M with the same involution k is a 2-morphism

$$\beta : \mathcal{A} \rightrightarrows \mathcal{A}'$$

which commutes with φ and φ' in the sense that the diagram

$$\begin{array}{ccc} k^* \mathcal{A} & \xrightarrow{\varphi} & \mathcal{A}^* \\ k^* \beta \downarrow & & \downarrow \beta^* \\ k^* \mathcal{A}' & \xrightarrow{\varphi'} & \mathcal{A}'^* \end{array}$$

is commutative.

Recall that 2-morphisms are certain isomorphisms of vector bundles, so that the diagram is in fact a diagram of isomorphisms of line bundles over Z . The definition of a morphism between Jandl structures allows us to consider the set of equivalence classes of Jandl structures on \mathcal{G} with involution k . Recall that by Proposition 2 the set of equivalence classes of isomorphisms is a torsor over the group of isomorphism classes of flat line bundles over M . The following theorem is a refinement for Jandl structures.

Theorem 2 ([SSW05]). *The set of equivalence classes of Jandl structures on a bundle gerbe \mathcal{G} with involution k is a torsor over the group of isomorphism classes of flat k -equivariant line bundles over M .*

This theorem can be used to determine the number of inequivalent choices of a Jandl structure, for instance for manifolds which have been considered before in unoriented string theories – so called orientifolds, e.g. [BPS92, BCW01, HSS02]. The following known results are reproduced.

- For $M = SU(2)$ with involution $k(g) := zg^{-1}$ for any element z in the center of G , and for a bundle gerbe \mathcal{G} over M whose curvature is an integral multiple of the 3-form H introduced in the introduction, there are two inequivalent Jandl structures on \mathcal{G} with respect to k .
- For $M = SO(3)$ with involution $k(g) = g^{-1}$ and a bundle gerbe \mathcal{G} which is isomorphic to $k^*\mathcal{G}^*$, there are four inequivalent Jandl structures on \mathcal{G} with respect to k .
- For the 2-torus $T = S^1 \times S^1$ with involution $k = \text{id}_T$ and for a bundle gerbe \mathcal{G} which admits Jandl structures, their equivalence classes are in bijection to the group $\mathbb{Z}_2 \times U(1) \times U(1)$.

Let us now use a Jandl structure on a bundle gerbe \mathcal{G} . We are going to pursue the idea of descent holonomy. The existence of the isomorphism \mathcal{A} assures that the holonomy locally coincides on both sheets – now we have to make local selections of one of them. This is exactly what a choice of a fundamental domain of Σ in $\hat{\Sigma}$ does. It can be constructed locally as shown in Figure 5 as a submanifold F of $\hat{\Sigma}$ with (piecewise smooth) boundary. A key observation, which can be heuristically seen from Figure 5, is that the involution σ restricts to an orientation-*preserving* involution on $\partial F \subset \hat{\Sigma}$. Accordingly, the quotient $\overline{\partial F}$ is an oriented closed submanifold of Σ .

Remember the following two situations: two trivializations of the same bundle gerbe define a line bundle over M with a certain curvature (Lemma 1). A trivialization together with a D-brane gives a vector bundle with a

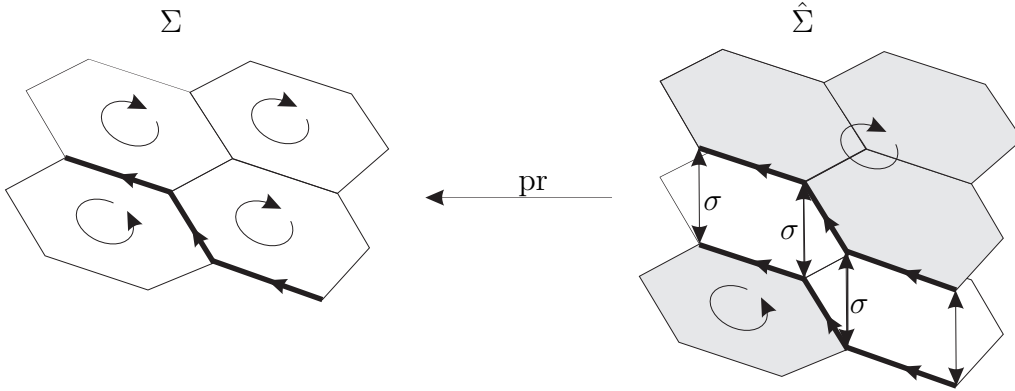


Figure 5: On the left hand side we show a dual triangulation, i.e. every vertex has three edges. According to a choice of local orientations for every face of the triangulation – which is always possible – select one of the two preimages of this face under the covering $\text{pr} : \hat{\Sigma} \rightarrow \Sigma$. This defines a fundamental domain as the grey-shaded surface on the right hand side.

certain curvature. A similar situation appears for a trivialization together with a Jandl structure.

Lemma 2 ([SSW05]). *A trivialization $\mathcal{T} : \mathcal{G} \rightarrow \mathcal{I}_\rho$ and a Jandl structure \mathcal{J} on \mathcal{G} with involution k determine a k -equivariant line bundle $R \rightarrow M$ with curvature $k^*\rho + \rho$.*

Now we are ready to put the pieces together:

Definition 10 ([SSW05]). *Let \mathcal{G} be a bundle gerbe over M with Jandl structure \mathcal{J} with involution k . Let Σ be a closed surface with orientation covering $\hat{\Sigma}$ and let $\hat{\phi} : \hat{\Sigma} \rightarrow M$ be an equivariant map. The pullback along $\hat{\phi}$ gives a bundle gerbe $\hat{\phi}^*\mathcal{G}$ over $\hat{\Sigma}$ with Jandl structure $\hat{\phi}^*\mathcal{J}$ with involution σ . A choice of a trivialization*

$$\mathcal{T} : \hat{\phi}^*\mathcal{G} \rightarrow \mathcal{I}_\rho$$

determines in combination with the Jandl structure a σ -equivariant line bundle $R \rightarrow \hat{\Sigma}$. In turn, this equivariant line bundle determines a quotient line bundle $\bar{R} \rightarrow \Sigma$. Let F be a fundamental domain. Then we define

$$\text{hol}_{\mathcal{G}, \varepsilon, \mathcal{J}}(\hat{\phi}) := \exp \left(\int_F \rho \right) \cdot \text{hol}_{\bar{R}}(\bar{\partial}F)$$

to be the holonomy of the bundle gerbe \mathcal{G} with Jandl structure \mathcal{J} around $\hat{\phi}$.

In [SSW05] we show that this definition depends neither on the choice of the trivialization nor on the choice of the fundamental domain.

To close, let us remark that also a Jandl structure can be understood in terms of local data. Recall that an isomorphism \mathcal{A} relates local data of bundle gerbes, here

$$-(g, A, B) = k^*(g, A, B) + D(t, W).$$

It is also possible to extract a function $j_i : V_i \rightarrow U(1)$ from the 2-morphism φ , which relates the local data (t, W) of the isomorphism \mathcal{A} to their pullback,

$$(t, W) = k^*(t, W) + D(j).$$

Finally, the condition on the 2-morphism φ leads to

$$j^{-1} = k^*j.$$

In [SSW05] an expression for the holonomy $\text{hol}_{\mathcal{G}, \mathcal{E}, \mathcal{J}}(\hat{\phi})$ is derived in terms of the local data of the bundle gerbe and of the Jandl structure, analogous to (2). We don't give the full expression here, but indicate how the local data should be placed on a triangulated surface in Figure 6.

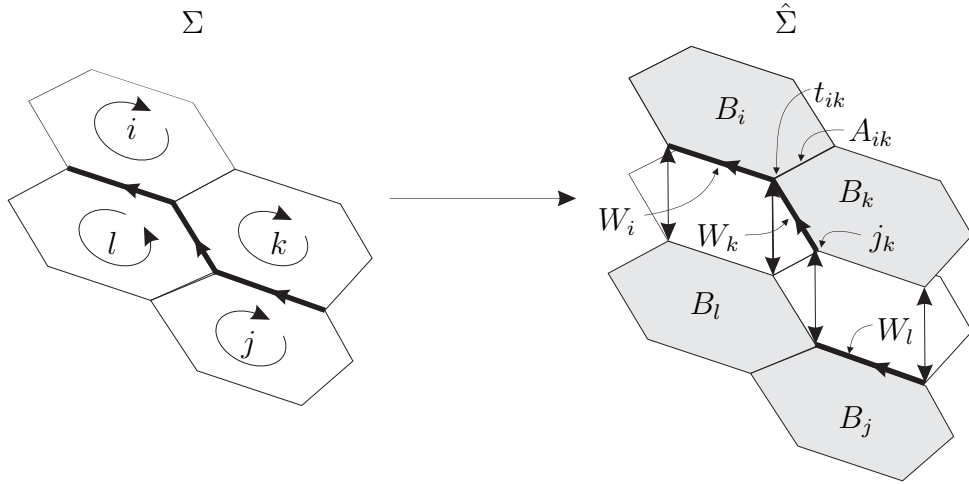


Figure 6: The oriented triangulation of the closed surface Σ determines a fundamental domain, which is decorated by the local data of the bundle gerbe in its interior and by the local data of the Jandl structure along the orientation reversing edges.

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